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Comments on Models of Scalar–Tensorial Field Equations in General Relativity

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Abstract

In this paper we consider some of the proposed models for introducing the long-range scalar interaction in Riemannian space-times. The relationship among these models is discussed. Particular emphasis is placed on the introduction of the scalar interaction via a conformal mapping on the original Riemannian geometry. Following this method we introduce a spinorial model for the coupled system. We also discuss the meaning of the identities satisfied by the left-hand side of the coupled field equations, which are present for any model derivable from an action principle.

1. *Introduction*

The introduction of a scalar field in the gravitational field theory of general relativity, due originally to C. Brans and R. H. Dicke (1961), was subsequently studied by several authors. Here we consider a general formulation due to G. Horndeski (1974), which besides considering the structure of general Legrange scalar densities depending on the metric tensor g_{ab}, a scalar field ϕ, and their derivatives of arbitrary order, also consider the most general action principle involving second-order Euler Lagrange equations. We show that this general formulation degenerates in Bergmann's formulation (1968) by means of a particular choice of the arbitrary functions of the scalar field which are present in the general Lagrangian density proposed by Horndeski.

The left-hand side of the Euler Lagrange equations for g_{ab} and ϕ, which are geometrical objects with the structure of a symmetric tensor density of weight + 1 and a scalar density of the same weight, satisfy four constraint relations. These constraint relations will appear for any formulation derivable from the action principle. Here we show that these constraints may be understood at least from two different arguments. The first of these arguments is a simple generalization to the Riemannian geometry of a similar result which holds in the Lorentz-covariant field theory involving the free scalar field.

In this paper we also propose some models of scalar–tensor theories derivable from a conformal transformation on the original Riemannian space-time describing the free gravitational field. We use this process for introducing a spinor formulation for the coupled system (g_{ab}, ϕ).

The notation used here is as follows: Lowercase Latin letters designate tensor indices running from 0 to 3. Capital Latin letters refer to spinor degrees of freedom, and run from 1 to 2 in the complex domain. The local signature of the metric tensor is −2. For any quantity B, the notation $B_{,a}$ indicates a partial derivative of B, and the notation $B_{|a}$ the covariant derivative of the quantity B.

2. *General Vacuum Field Equations of Scalar–Tensor Theories in Four-Dimensional Riemannian Spaces*

In this section we make some considerations on the general formulation proposed by Horndeski. Consider a general Lagrange scalar density that is a concomitant of a pseudo-Riemannian metric tensor, a scalar field ϕ, and their derivatives of arbitrary order:

$$\mathscr{L}_H = \mathscr{L}_H(g_{ab}; g_{ab,i_1}; \ldots g_{ab,i_1 \ldots i_p}; \phi; \phi_{,i_1} \ldots \phi_{,i_1 \cdots i_q})$$

for $p, q \geqslant 2$. The Euler-Lagrange tensor densities corresponding to this Lagrangian are

$$E^{ab} = \sum_{h=0}^{p} (-1)^{h+1} \frac{\partial}{\partial x^{i_1}} \cdots \frac{\partial}{\partial x^{i_h}} \frac{\partial \mathscr{L}_H}{\partial g_{ab,i_1 \cdots i_h}} \tag{2.1}$$

$$E = \sum_{h=0}^{q} (-1)^{h+1} \frac{\partial}{\partial x^{i_1}} \cdots \frac{\partial}{\partial x^{i_h}} \frac{\partial \mathscr{L}_H}{\partial \phi_{,i_1 \cdots i_h}} \tag{2.2}$$

From a mathematical technique due to du Plessis (1969), it is possible to show that E^{ab} and E are not independently defined, but are related by the four constraint conditions (Horndeski and Lovelock, 1972; Horndeski, 1971)

$$E^{ab}{}_{|b} = \tfrac{1}{2}\phi^{|a}E \tag{2.3}$$

This result shows that the Euler-Lagrange equation for the scalar field is a consequence of the tensorial equations. Indeed, defining

$$\rho = g^{ab}\phi_{|a}\phi_{|b} = g^{ab}\phi_{,a}\phi_{,b} \tag{2.4}$$

we have

$$E = (2/\rho)\phi_{|a}E^{ab}{}_{|b}$$

Similar relations can also be derived for other types of interactions with the gravitational field, as for instance, for the Einstein–Maxwell theory (Lovelock, 1973).

The usual results of field theory apply for second-order equations; therefore, we are more directly interested in the situation where $p = q = 2$. All the subsequent discussion will refer to this situation. The origin of the identity relation

(2.3) may be drawn to a result of the conventional Lorentz-covariant field theory of a free real one-component field ϕ (Schweber et al. 1955), with Lagrangian density $\overset{(0)}{\mathscr{L}}$:

$$\frac{\partial \overset{(0)}{T}{}^{ab}}{\partial x^b} = \phi^{,a}\left(\frac{\partial}{\partial x^b}\frac{\partial \overset{(0)}{\mathscr{L}}}{\partial \phi_{,b}} - \frac{\partial \overset{(0)}{\mathscr{L}}}{\partial \phi}\right) \tag{2.5}$$

the generalization of this formula to a Riemannian space-time being (we indicate by $\mathscr{L}$ the new Lagrangian density)

$$(\sqrt{-g}T^{ab})_{|b} = \phi^{|a}E(\mathscr{L}), \qquad E(\mathscr{L}) = \sqrt{-g}\,\mathscr{E}(\mathscr{L}) \tag{2.6}$$

Recalling that the Einstein's tensor G^{ab} satisfies the four contracted Bianchi identities

$$G^{ab}{}_{|b} = 0 \tag{2.7}$$

And that the gravitational field equations coupled to a scalar field are of the form

$$E^{ab} \equiv \sqrt{-g}(G^{ab} + T^{ab}) = 0 \tag{2.8}$$

we have from (2.7) and (2.8)

$$E^{ab}{}_{|b} = \sqrt{-g}(G^{ab}{}_{|b} + T^{ab}{}_{|b}) = (\sqrt{-g}\,T^{ab})_{|b}$$

From this relation and from (2.6) we get a formula similar to (2.3). The factor 1/2 which will be absent in this formula may be introduced if we define the Lagrangian with a multiplicative factor 1/2. In this case $\overset{(0)}{T}{}^{ab}$ is also defined with the same constant multiplicative factor.

Horndeski proposed the following Lagrange scalar density (for $q = p = 2$) for the coupled system (g_{ab}, ϕ):

$$\begin{aligned}\mathscr{L}_H = {} & \tfrac{1}{4}\sqrt{-g}\,\beta_1\,\delta^{abcd}_{efhi}R_{ab}{}^{ef}R_{cd}{}^{hi} \\ & -\tfrac{1}{4}\sqrt{-g}\beta_2\delta^{abc}_{def}\phi_{|a}\phi^{|d}R_{bc}{}^{ef} + \tfrac{1}{2}\sqrt{-g}\beta_3\delta^{ab}_{cd}R_{ab}{}^{cd} \\ & + \sqrt{-g}\eta + C\epsilon^{abcd}R^p{}_{qab}R^q{}_{pcd}\end{aligned} \tag{2.9}$$

where β_1, β_2, and β_3 are arbitrary functions of the scalar field ϕ. These functions are assumed to be independent of the g_{ab}. The quantity η is an arbitrary function of ϕ and ρ; and finally, C is a constant. The first and last terms in this Lagrangian density may be written as

$$\sqrt{-g}\,[(R^2 - \tfrac{1}{4}R_{ab}R^{ab} + R_{abcd}R^{abcd})\beta_1 + C\epsilon^{abcd}R^p{}_{qab}R^q{}_{pcd}]$$

some of these terms have, formally, a structure similar to the gauge invariant terms $F_{ab}F^{ab}$ and $\epsilon^{abcd}F_{ab}F_{cd}$ of the Maxwell Lagrangian density for the electromagnetic field. Owing to this analogy we may call the factors associated to $\beta_1(\phi)$ and to C the terms in $\mathscr{L}_H$ corresponding to a "gauge invariant formulation." However, we should note that the terminology "gauge invariant," used in this formal correspondence, is not to be understood as a true gauge-invariant formulation. This sort of theory is obtained only for the linear approximation to the field equations, and clearly this is not the case here. The second term on the right-hand side of (2.9) is an interaction term not satisfying the requirements of a minimal coupling. The third term is the usual term that generates the left-hand side of Einstein's equations. Finally, the term $\sqrt{-g}\eta(\phi, \rho)$ is a general expression involving the field $\phi(x)$ and the quantity $\rho(x)$. In the absence of the gravitational field $\mathscr{L}_H$ has the simplified form

$$\mathscr{L}_H = \eta(\phi, \overset{(0)}{\rho})$$

$$\overset{(0)}{\rho} = \eta^{ab}\phi_{,a}\phi_{,b}$$

The particular choice

$$\eta(\phi, \overset{(0)}{\rho}) = F(\phi) + G(\overset{(0)}{\rho}) = \tfrac{1}{2}\mu^2\phi^2 + \tfrac{1}{2}\overset{(0)}{\rho}$$

generates the Klein-Gordon equation for a scalar meson with rest mass μ.

In the remaining of this work we shall use a simplified form for $\mathscr{L}_H$ obtained by taking

$$\begin{aligned} \beta_1(\phi) &= \beta_2(\phi) = 0 \\ \beta_3(\phi) &= f_1(\phi) \\ C &= 0 \end{aligned} \tag{2.10}$$

And the function $\eta(\phi, \rho)$ is expanded in Taylor series on the quantity $\rho(x)$ retaining only the linear term

$$\eta(\phi, \rho) = f_3(\phi) + \rho f_2(\phi) \tag{2.11}$$

The reason for considering this particular expression for η is due to the fact that we want to obtain a close relationship with the usual field theory formulation of the Klein-Gordon equation [for flat space-times the choices $f_3(\phi) = \frac{1}{2}\mu^2\phi^2$, $f_2(\phi) = \frac{1}{2}$ generate such an equation].

The Lagrangian density that emerges as a consequence of the choices (2.10) and (2.11) is

$$\mathscr{L}_B = \sqrt{-g}(f_1 R + f_2\rho + f_3)$$

This Lagrangian density was proposed by Bergmann (1968). For this Lagrangian the rest mass term for the scalar field in the flat space-time limit arises from the cosmological term $\sqrt{-g}f_3$(for the previous choices of f_3 and f_2). The field equations derivable from $\mathscr{L}_B$ are

$$\begin{aligned} E^{ab}(\mathscr{L}_B) \equiv f_1\sqrt{-g}\,G^{ab} + \sqrt{-g}f_1'\,(g^{ab}\phi_{|e}{}^{|e} - \phi^{|ba}) \\ + \sqrt{-g}\,[\rho f_1'' - \tfrac{1}{2}(f_3 + \rho f_2)]g^{ab} + \sqrt{-g}(f_2 - f_1'')\phi^{|a}\phi^{|b} = 0 \end{aligned} \tag{2.12}$$

$$E(\mathscr{L}_B) \equiv -\sqrt{-g}\,[f_1' R - \rho f_2' + f_3' - 2f_2 g^{ab}\phi_{|ab}] = 0 \tag{2.13}$$

It may be seen from a direct calculation that the left-hand side of the equations (2.12) and (2.13) satisfy the conditions (2.3). In (2.12) and (2.13) we used the notation

$$f_i' = \frac{\partial f_i}{\partial \phi}$$

3. *Introduction of the Scalar Interaction by Means of a Conformal Transformation*

As is well known, the scalar interaction may be introduced by means of a conformal transformation on the metric tensor g_{ab}. For the free gravitational field we have

$$\mathscr{L}_0 = \sqrt{-g}R$$

$$G^{ab} = \frac{\delta \mathscr{L}_0}{\delta g_{ab}}$$

A conformal transformation on g_{ab} of the form

$$\tilde{g}_{ab} = e^{2\sigma(x)} g_{ab} \tag{3.1}$$

induces on the Ricci scalar the following variation (Eisenhart, 1960):

$$\tilde{R} = e^{-2\sigma}(R + 6\Delta_1\sigma + 6\Delta_2\sigma) \tag{3.2}$$

where

$$\Delta_1\sigma = g^{ab}\sigma_{|a}\sigma_{|b} = \alpha$$

$$\Delta_2\sigma = g^{ab}\sigma_{|ab} = \Box\sigma$$

We take for the Lagrangian density of the coupled system (g_{ab}, ϕ) the expression for the conformal variation of the free Lagrangian $\mathscr{L}_0$:

$$\mathscr{L} = \sqrt{-\tilde{g}}\tilde{R} = \sqrt{-g}e^{2\sigma}(R + 6\Box\sigma + 6\alpha)$$

recalling that

$$\Box\sigma = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^a} (\sqrt{-g}\sigma^{|a})$$

we obtain by partial integration, neglecting a surface term

$$I = \int \mathscr{L} d_4x$$

$$\mathscr{L} = \sqrt{-g}\varphi(\sigma)[R - 6\alpha] \tag{3.3}$$

where,

$$\varphi(\sigma) = e^{2\sigma}$$

Taking $\sigma = \sigma(\phi)$, we get

$$\alpha = \sigma'^2 \rho, \qquad \varphi(\phi) = e^{2\sigma(\phi)}$$

and the equation (3.3) takes the form

$$\mathscr{L} = \sqrt{-g}\,\varphi(\phi)[R - 6\sigma'^2 \rho] \tag{3.4}$$

which is a particular case of the Lagrangian density $\mathscr{L}_B$ for

$$f_1(\phi) = \varphi(\phi)$$

$$f_2(\phi) = -6\sigma'^2 \varphi(\phi)$$

$$f_3(\phi) = 0$$

In the absence of the gravitational field the Lagrangian density (3.4) describes a scalar field with variable rest mass. Indeed, the field equations obtained from (3.4) have the form

$$E^{ab} \equiv \sqrt{-g}\,\varphi(\phi)[G^{ab} + 2\sigma'(g^{ab}\phi_{|e}{}^{|e} - \phi^{|ba}) + \rho g^{ab}(7\sigma'^2 + 2\sigma'') - 2\phi^{|a}\phi^{|b}(\sigma'' + 5\sigma'^2)] = 0 \tag{3.5}$$

$$E \equiv 2\sqrt{-g}\,\sigma'\varphi(\phi)[-R - 6\rho(\sigma'^2 + \sigma'') - 6\sigma'\Box\phi] = 0 \tag{3.6}$$

The field equation (3.6), for the scalar field, in the limit where $R \to 0$, $g_{ab} \to \eta_{ab}$ takes the form

$$\underset{(0)}{E} \equiv 12\sigma'^2\varphi(\phi)\left[-\frac{\sigma'^2 + \sigma''}{\sigma'}\underset{(0)}{\rho} - \underset{(0)}{\Box}\phi\right] = 0$$

where $\underset{(0)}{\Box}$ is the "flat D'Alembertian operator"

$$\underset{(0)}{\Box} = \eta^{ab}\partial_a\partial_b$$

This equation has a mass term of the form

$$m^2(\phi) = \frac{\sigma'^2 + \sigma''}{\phi\sigma'}\,\eta^{ab}\phi_{,a}\phi_{,b}$$

The present type of approach allows us to introduce the interaction of the gravitational field with several scalar fields, for instance, for two of such fields we may associate the following conformal transformation on the components of the metric tensor;

$$\tilde{g}_{ab} = e^{2(\sigma_1 + \sigma_2)} g_{ab}$$

writing

$$\sigma = \sigma_1 + \sigma_2$$

we get a total Lagrangian density given by (3.3) for

$$\varphi(\sigma) = e^{2[\sigma_1(\phi_1) + \sigma_2(\phi_2)]} = \varphi_1(\phi_1)\varphi_2(\phi_2)$$

Then,

$$\alpha = g^{ab}(\sigma_1 + \sigma_2)_{|a}(\sigma_1 + \sigma_2)_{|b} = \alpha_1 + \alpha_2 + 2\sigma_1'\sigma_2'\phi_{1|a}\phi_2^{|a}$$

where

$$\alpha_1 = g^{ab}\sigma_{1|a}\sigma_{1|b} = \sigma_1'^2\rho_1$$
$$\alpha_2 = g^{ab}\sigma_{2|a}\sigma_{2|b} = \sigma_2'^2\rho_2$$

Thus, the Eq. (3.3) assumes the form

$$\mathscr{L}_{12} = \sqrt{-g}\varphi_1(\phi_1)\varphi_2(\phi_2)\,[R - 6\sigma_1'^2\rho_1 - 6\sigma_2'^2\rho_2 - 12\sigma_1'\sigma_2'\phi_{1|a}\phi_2^{|a}] \quad (3.7)$$

This Lagrangian density may describe the interaction of the gravitational field with a charged scalar field. We may write

$$\mathscr{L}_{12} = \sqrt{-g}\,\varphi_1\varphi_2[R - 6g^{ab}\underset{(1,2)}{\psi_{ab}}(\phi_i, \phi_{i|c})] \quad (3.8)$$

(the index i in the scalar quantity is just an abbreviation and runs from 1 to 2). We have

$$\underset{(1,2)}{\psi_{ab}} = \sigma_1'^2\phi_{1|a}\phi_{1|b} + \sigma_2'^2\phi_{2|a}\phi_{2|b} + \sigma_1'\sigma_2'(\phi_{1|a}\phi_{2|b} + \phi_{1|b}\phi_{2|a}) \quad (3.9)$$

The expression for $\mathscr{L}_{12}$ is of the same general structure as the expression for the Lagrangian corresponding to one-component scalar interaction. Indeed, from (3.4) we have

$$\mathscr{L} = \sqrt{-g}\,\varphi[R - 6g^{ab}\psi_{ab}(\phi, \phi_{|c})] \quad (3.10)$$

$$\psi_{ab} = \sigma'^2\phi_{|a}\phi_{|b} \quad (3.11)$$

Therefore, for variations on the g_{ab}, for fixed ϕ_1 and ϕ_2, we just replace in the corresponding variation in (3.10) φ by $\varphi_1\varphi_2$ and ψ_{ab} by $\underset{(1,2)}{\psi_{ab}}$. Since the variation on the g_{ab} in (3.10) has the form

$$\delta\int\mathscr{L}\,d_4x = \int G_{ab}\delta g^{ab}\varphi\sqrt{-g}d_4x + \int g^{ab}\varphi\sqrt{-g}\delta R_{ab}d_4x - 6\int\sqrt{-g}\varphi\psi_{ab}\delta g^{ab}d_4x$$
$$+ 3\int\sqrt{-g}\,\varphi g^{rs}\psi_{rs}g_{ab}\delta g^{ab}d_4x \quad (3.12)$$

Where the second term on the right-hand side may be written as (Landau and Lifschitz, 1951)

$$\int g^{ab}\varphi\sqrt{-g}\delta R_{ab}d_4x = \int\varphi\frac{\partial}{\partial x^a}(\sqrt{-g}\delta w^a)d_4x \quad (3.13)$$

for

$$\delta w^a = g^{mn}\delta\Gamma^a_{mn} - g^{ma}\delta\Gamma^n_{mn}$$

Using this expression a long but otherwise straightforward calculation gives (neglecting surface terms)

$$\int g^{ab}\varphi\sqrt{-g}\,\delta R_{ab}d_4x = \int \sqrt{-g}\,\{g^{mn}(g^{bc}\varphi_{|c})_{|m} - g^{ab}(g^{mc}\varphi_{|c})_{|m}\}\delta g_{ab}d_4x \tag{3.14}$$

Therefore

$$\delta \int \mathscr{L} d_4x = \int \sqrt{-g}[-G^{ab}\varphi - g^{ab}(g^{mc}\varphi_{|c})_{|m} + g^{ma}(g^{bc}\varphi_{|c})_{|m} + 6\varphi\psi^{ab} - 3\varphi g^{rs}\psi_{rs}g^{ab}]\,\delta g_{ab}d_4x$$

giving

$$E^{ab} \equiv \sqrt{-g}[\varphi G^{ab} + g^{ab}(g^{mc}\varphi_{|c})_{|m} - g^{ma}(g^{bc}\varphi_{|c})_{|m} - 6\varphi\psi^{ab} + 3\varphi g^{rs}\psi_{rs}g^{ab}] \tag{3.15}$$

Substituting $\varphi(\phi)$ by $\exp[2\sigma(\phi)]$ in this expression we get the previous equation (3.5). The corresponding equation for the system with two scalar fields is obtained by the replacements

$$\varphi \to \varphi_1\varphi_2$$
$$\psi_{ab} \to \underset{(1,2)}{\psi_{ab}}$$

Thus,

$$\begin{aligned}\underset{(1,2)}{E^{ab}} \equiv \sqrt{-g}\varphi_1\varphi_2[G^{ab} &+ 2\sigma_1'(g^{ab}\Box\phi_1 - \phi_1^{|ba}) \\ &+ 2\sigma_2'(g^{ab}\Box\phi_2 - \phi_2^{|ba}) + \rho_1 g^{ab}(7\sigma_1'^2 + 2\sigma_1'') \\ &+ \rho_2 g^{ab}(7\sigma_2'^2 + 2\sigma_2'') - 2\phi_1^{|a}\phi_1^{|b}(\sigma_1'' + 5\sigma_1'^2) \\ &- 2\phi_2^{|a}\phi_2^{|b}(\sigma_2'' + 5\sigma_2'^2) + 2\sigma_1'\sigma_2'(7g^{ab}\phi_{1|c}\phi_2^{|c} \\ &- 5\phi_1^{|a}\phi_2^{|b} - 5\phi_1^{|b}\phi_2^{|a})]\end{aligned} \tag{3.16}$$

In the limit where $\phi_1 \to \phi$, $\phi_2 \to 0$[1] (or the reverse), we re-obtain the equation (3.5). The left-hand side of the field equation for ϕ_1 has the form

$$\begin{aligned}E_1\{\phi_1\} \equiv 2\sqrt{-g}\varphi_1\varphi_2\sigma_1'[-R &- 6\rho_1(\sigma_1'' + \sigma_1'^2) - 6\sigma_1'\Box\phi_1 \\ &- 12\phi_{1|a}\phi_2^{|a}\sigma_1'\sigma_2' - 6\rho_2(\sigma_2'' + \sigma_2'^2) - 6\sigma_2'\Box\phi_2]\end{aligned} \tag{3.17}$$

For obtaining the corresponding left-hand side of the field equation for ϕ_2 just replace the subscripts "1" by "2" and "2" by "1." The equation (3.17) goes over

[1] For $\phi_2 \to 0$, we have that $\varphi_2 \to e^{2\sigma_2(0)}$ = const, assuming that $\sigma_2(\phi_2)$ is finite at the origin. This constant factor may be absorbed in the function $\varphi_1(\phi_1) = \varphi(\phi)$.

(3.6) in the limit where $\phi_1 \to \phi$, $\phi_2 \to 0$ [the same consideration of a common constant multiplicative factor arises here, similarly to the situation for the tensorial equation (3.16)]. It should be noted that, in spite of looking similar, the quantities $E_1\{\phi_1\}$ and $E_2\{\phi_2\}$ generate field equations that are independent one from the other. Indeed, the field equations for ϕ_1 and ϕ_2 may be put as

$$-6\sigma_1'^2\Box\phi_1 - 6\rho_1\sigma_1'(\sigma_1'' + \sigma_1'^2) - R\sigma_1' = J_{2\to 1}$$
$$-6\sigma_2'\Box\phi_2 - 6\rho_2\sigma_2'(\sigma_2'' + \sigma_2'^2) - R\sigma_2' = J_{1\to 2}$$

where $J_{i\to k}$ are the corresponding source terms for the coupling between the fields ϕ_1 and ϕ_2 in the presence of the gravitational field:

$$J_{i\to k} = 12\phi_{i|a}\phi_k^{|a}\sigma_i'\sigma_k' + 6\rho_i(\sigma_i'' + \sigma_i'^2) + 6\sigma_i'\Box\phi_i$$

The identities relating the left-hand side of the field equations now have presently the form

$$\underset{(1,2)}{E}{}^{ab}{}_{|b} = \tfrac{1}{2}\phi_1^{|a}E_1\{\phi_1\} + \tfrac{1}{2}\phi_2^{|a}E_2\{\phi_2\}$$

This process may be generalized in order to include more scalar components. For instance, introducing a third scalar field in this method is equivalent to considering the coupling among a charged meson, a neutral meson, and the underlying gravitational field.

4. *The Scalar Interaction in the Spinor Formulation*

In the present work we consider only the interaction among scalar fields and the underlying gravitational field. Both systems being bosonic in their structure do not need a spinor formulation for the mathematical content of the formulas describing the interaction. However, from the viewpoint of elementary particles and fields, if we want to work with fermionic systems in this scheme we will need the spinor formulation for the correct, mathematical, formulation of the coupled system. For instance, consider the problem of considering the coupling between the Brans-Dicke field (that means the system g_{ab}, ϕ) and a system of fermionic fields. For treating this complex system we need a spinor formalism. Since in general this fermionic system possesses some internal symmetry properties, we may ask whether such symmetries are kept unchanged under the constraint that relates the left-hand sides of the several field equations for the coupled system. It should be noted that these types of questions are more naturally treated in a quantized theory for the whole system. However, since we do not have at present such a theory, we may consider the problem on the classical level.

In what follows we treat the mathematical problem of translating the present tensorial formulation for the coupled (g_{ab}, ϕ) system in a spinorial language. This method serves as a first step towards the more general problem referred to previously. First we introduce some general results concerning the spinor formalism.

Associated with an everywhere regular region G of the four-dimensional

Riemannian space there exists a two-dimensional symplectic space, that is, a linear vector space over the field of the complex numbers equipped with a nondegenerate skew symmetric bilinear scalar product. The connection between geometrical objects defined on points $x^b \in G$ with geometrical objects belonging to the symplectic space is obtained via a set of four Hermitian 2×2 matrices $\sigma_a = \sigma_a{}^{A\dot{M}}(x)$. Spinor indices will be denoted by capital Latin letters. Denoting by S_2 the two-dimensional symplectic space corresponding to G, the objects σ_a possess the index a belonging to G and the pair of indices $A, \dot{M}$ belonging to S_2. As an example, to a vector field on G, say A_a, corresponds a second-rank spinor $V_{A\dot{M}}$ on S_2,

$$V_a = \sigma_a{}^{A\dot{M}}(x) V_{A\dot{M}}(x)$$

The following definitions for raising and lowering spinor indices are used:

$$u^A = \epsilon^{AB} u_B = -u_B \epsilon^{BA}$$

$$u^A = u^B \epsilon_{BA} = -\epsilon_{AB} u^B$$

where $\epsilon = (\epsilon^{AB})$ is the skew symmetric metric on S_2,

$$\epsilon^{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \epsilon^{\dot{A}\dot{B}}$$

We define the set of 2×2 Hermitian matrices

$$\tau_a = \epsilon \bar{\sigma}_a \epsilon \tag{4.1}$$

where a bar means complex conjugation. This relation in the index notation takes the form

$$\tau_{a N\dot{M}} = -\bar{\sigma}_{a N\dot{M}} = -\overset{T}{\sigma}_{a N\dot{M}} \tag{4.2}$$

With the set of Hermitian matrices σ_a and τ_a we can define the metric tensor g_{ab} on G by means of

$$\sigma_a \tau_b + \sigma_b \tau_a = -2 g_{ab} \cdot I \tag{4.3}$$

where I is the 2×2 identity matrix. This equation in the index notation reads

$$\sigma_a{}^{K\dot{M}} \sigma_{b \dot{M}R} + \sigma_b{}^{K\dot{M}} \sigma_{a \dot{M}R} = 2 g_{ab} \delta_R{}^K \tag{4.4}$$

We may write

$$g_{ab} = -\tfrac{1}{2} \operatorname{Tr}(\sigma_a \tau_b) \tag{4.5}$$

The internal space S_2 possess a connection $\Gamma_a{}^A{}_B$ and a curvature $P_{ab}{}^A{}_B$, and we have the usual relations:

$$u^A{}_{|ab} - u^A{}_{|ba} = P_{ab}{}^A{}_B u^B$$

$$P_{ab} = \Gamma_{a,b} - \Gamma_{b,a} - \Gamma_a \Gamma_b + \Gamma_b \Gamma_a$$

The components of the Riemannian curvature are related to the components of P_{ab} by

$$P_{ab} = \tfrac{1}{4}\tau^c\sigma^d R_{abcd}$$

The inverse relations are also known in the extensive literature on this subject.

We consider now a conformal transformation on the set of matrices σ_a and τ_a

$$\begin{aligned} \tilde{\sigma}_a &= e^{\lambda}\sigma_a \\ \tilde{\tau}_a &= e^{\lambda}\tau_a \end{aligned} \tag{4.6}$$

where λ is a real scalar. This transformation induces on the g_{ab} a conformal variation similar to the variation considered in equation (3.1). Thus, we may write

$$\sqrt{-\tilde{g}}\tilde{R} = \sqrt{-g}\varphi(\lambda)[R + 6\Box\lambda + 6\beta] \tag{4.7}$$

for

$$\begin{aligned} \varphi(\lambda) &= e^{2\lambda} \\ \beta &= g^{ab}\lambda_{|a}\lambda_{|b} \end{aligned}$$

As before $\sqrt{-g}R$ is considered as the Lagrangian density for the free gravitational field, and $\sqrt{-\tilde{g}}\tilde{R}$ is the Lagrangian density for the coupled system (g_{ab}, λ). By partial integration neglecting the surface term we find

$$\mathscr{L} = \sqrt{-g}\,\varphi(\lambda)[R - 6\beta] \tag{4.8}$$

First of all we determine the tensorial equation for a free gravitational field in this spinor formalism. Starting with

$$\underset{(0)}{\mathscr{L}} = \sqrt{-g}R$$

and taking variations on the σ_a, using (4.1) and (4.5) we get

$$\begin{aligned} \delta\underset{(0)}{\mathscr{L}} &= -\sqrt{-g}G^{ab}\delta g_{ab} = \tfrac{1}{2}\sqrt{-g}G^{ab}\mathrm{Tr}[(\delta_a{}^r\tau_b + \delta_b{}^r\tau_a)\delta\sigma_r] \\ &= \tfrac{1}{2}\sqrt{-g}\ \mathrm{Tr}[(G^{rb}\tau_b + G^{ra}\tau_a)\delta\sigma_r] \end{aligned}$$

since G^{ab} is symmetric, we obtain simply

$$\delta\underset{(0)}{\mathscr{L}} = \sqrt{-g}\ \mathrm{Tr}(G^{ar}\tau_a\delta\sigma_r)$$

Therefore, the equations for the free gravitational field have the form

$$\underset{(0)}{E^r} \equiv \sqrt{-g}G^{ar}\tau_a = 0 \tag{4.9}$$

In the index notation they assume the form

$$\underset{(0)}{E^r{}_{M\dot{N}}} \equiv -\sqrt{-g}G^{ar}\sigma_{aM\dot{N}} = 0$$

In this case the contracted Bianchi identities imply that

$$\underset{(0)}{E}{}^{r}{}_{M\dot{N}|r} = -\sqrt{-g}G^{ar}\sigma_{aM\dot{N}|r}$$

From (4.3) recalling that g_{ab} has a null covariant derivative we get

$$\sigma_{a|r} = \tau_{a|r} = 0$$

Thus, in the present case the contracted Bianchi identities read

$$\underset{(0)}{E}{}^{r}{}_{M\dot{N}|r} = 0 \tag{4.10}$$

From the Lagrangian density (4.8) we get

$$\delta\mathscr{L} = -\sqrt{-g}\varphi G^{ab}\delta g_{ab} + \varphi(\sqrt{-g}\delta w^{a})_{|a}$$
$$- 3\sqrt{-g}\beta\varphi g^{ab}\delta g_{ab} + 6\sqrt{-g}\varphi\lambda^{|a}\lambda^{|b}\delta g_{ab}$$

[in this variation the field $\lambda(x)$ is kept fixed]. Using the previous expression for δw^{a}, we can write $\delta\mathscr{L}$ as (neglecting a divergence)

$$\delta\mathscr{L} = \sqrt{-g}\,[-G^{ab}\varphi - g^{ab}(g^{mc}\varphi_{|c})_{|m} + g^{ma}(g^{bc}\varphi_{|c})_{|m} + 6\varphi\lambda^{|a}\lambda^{|b} - 3\varphi\beta g^{ab}]\,\delta g_{ab}$$

or equivalently as

$$\delta\mathscr{L} = \tfrac{1}{2}\sqrt{-g}\,\{\varphi G^{ab} + g^{ab}(g^{mc}\varphi_{|c})_{|m} - g^{ma}(g^{bc}\varphi_{|c})_{|m}$$
$$- 6\varphi\lambda^{|a}\lambda^{|b} + 3\varphi\beta g^{ab}\}\mathrm{Tr}\,[(\delta_a^{\,r}\tau_b + \delta_b^{\,r}\tau_a)\delta\sigma_r] \tag{4.11}$$

The factor multiplying the trace in (4.11) is symmetric over the indices a, b. Denoting this factor by $\frac{1}{2}\sqrt{-g}\;\mathscr{E}^{ab}$, we have

$$\delta\mathscr{L} = \sqrt{-g}\,\mathrm{Tr}\,(\mathscr{E}^{ra}\tau_a\delta\sigma_r)$$

Therefore, the field equations for the gravitational field interacting with the scalar field $\lambda(x)$ are

$$E^{r} \equiv \sqrt{-g}\varphi(\lambda)[G^{ra} + 7\beta g^{ra} - 10\lambda^{|r}\lambda^{|a} - 2\lambda^{|ra} + 2g^{ra}\Box\lambda]\,\tau_a = 0 \tag{4.12}$$

For the case where $\lambda \to \sigma(\phi)$, and consequently $\beta \to \sigma'^{2}\rho$, this equation takes the form

$$E^{r} \equiv E^{ra}\tau_a$$

where E^{ra} is given by (3.5). By analogy with the formula (4.5) we will write the scalar quantity $\lambda(x)$ under the form

$$\lambda = -\tfrac{1}{2}\,\mathrm{Tr}(\omega\cdot\Omega) \tag{4.13}$$

where ω is a non-singular Hermitian 2 x 2 matrix, and

$$\Omega = \epsilon\bar{\omega}\epsilon = \Omega^{+}$$

It is easy to prove that the λ given by (4.13) is a real quantity. We may also prove that

$$\delta\lambda = -\mathrm{Tr}(\Omega \cdot \delta\omega) \tag{4.14}$$

In the index notation, the equation (4.13) reads as follows:

$$\lambda = \tfrac{1}{2}\omega_{A\dot{B}}\bar{\omega}^{\dot{B}A}$$

since $\Omega^{\dot{B}A} = -\bar{\omega}^{\dot{B}A} = -\overset{T}{\omega}{}^{\dot{B}A}$. The second-rank Hermitian spinor ω now describes the scalar field, similarly to the σ_a which describe the gravitational field according to the relation (4.5).

The Lagrangian density (4.8) under variation on the quantity $\lambda(x)$ becomes (neglecting the surface term)

$$\delta\mathscr{L} = \sqrt{-g}\varphi(\lambda)[2R - 12\beta]\,\delta\lambda + 12(\sqrt{-g}\varphi g^{ab}\lambda_{|b})_{,a}\delta\lambda$$

Since the second factor on the right-hand side, multiplying $\delta\lambda$, is a vector density of weight + 1, the partial derivative is equal to the covariant derivative, and we get

$$\delta\mathscr{L} = \sqrt{-g}\varphi(\lambda)[2R + 12\beta + 12\Box\lambda]\,\delta\lambda$$

Thus, the left-hand side of the field equation for the scalar field, in this spinor formulation, is given by

$$\delta\mathscr{L} = -\sqrt{-g}\varphi(\lambda)[2R + 12\beta + 12\Box\lambda]\,\mathrm{Tr}(\Omega \cdot \delta\omega)$$

and has the form

$$E \equiv -2\sqrt{-g}\varphi(\lambda)[R + 6\beta + 6\Box\lambda]\,\Omega \tag{4.15}$$

The geometrical objects given by (4.12) and (4.15) are, respectively, a vector and a scalar on G, and five Hermitian matrices on S_2.

Recalling the unitary five-dimensional field theory of Kaluza-Klein-Thirry we may use the notation $\Omega = \tau_5$. However, here we are not directly interested in this analogy, and we continue to use the symbol Ω.

The identity satisfied by the geometrical objects $E_{\dot{B}A}$ and $E^r{}_{\dot{B}A}$ now takes the form

$$E^r{}_{\dot{B}A\,|r} = \tfrac{1}{2}\lambda^{|a}\mathscr{E}\tau_{a\,\dot{B}A}$$

where

$$E_{\dot{B}A} = \mathscr{E}\,\Omega_{\dot{B}A}$$

$\mathscr{E}$ is the scalar differential given as the "nucleus" of equation (4.15).

5. *Concluding Remarks*

In this work we have considered some of the proposed models for the introduction of a scalar field in general relativity. The relationship among these models was studied. Presently we want to turn back again to the problem of existence of the four identities connecting the left-hand sides of the Euler-Lagrange equations for the scalar-tensorial system. We have seen in Section 1 that these identities may be obtained by a covariant generalization of a formula holding in the Lorentz-covariant field theory of a scalar field. Another way of understanding the existence of these identities is to recall that in the Lagrangian formulation of general relativity, the existence of the four contracted Bianchi identities implies that only six equations of motion are actually independent. This result, which is mathematically equivalent to the statement of general covariance of the theory is also verified for the scalar-tensorial theory.

As a final remark we note that from the general form of the tensorial equation coupled to a scalar field, we have

$$E^{ab} = G^{ab} + T^{ab} = 0$$

We have, as result of the existence of the four identities relating this system,

$$T^{ab}{}_{|b} = \tfrac{1}{2}\phi^{|a}E$$

Thus, the equations of the gravitational field contain the equation of motion of the scalar field which represents, in this case, the source for the gravitational field.

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